

# Proof of the Dual Conformal Anomaly of One-Loop Amplitudes in $\mathcal{N} = 4$ SYM

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## Abstract

We provide two derivations of the one-loop dual conformal anomaly of generic  $n$ -point superamplitudes in maximally supersymmetric Yang-Mills theory. Our proofs are based on simple applications of unitarity, and the known analytic properties of the amplitudes.

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# 1 Introduction

A novel symmetry of the planar S-matrix of  $\mathcal{N} = 4$  supersymmetric Yang-Mills (SYM) – dual superconformal symmetry – has been introduced in [1]. There, it was conjectured to be an exact symmetry at tree level, but broken by quantum corrections, and an expression for the anomaly associated to the dual conformal generators was proposed. A confirmation of the conjecture was presented shortly after in [2], where it was demonstrated that the tree-level S-matrix of  $\mathcal{N} = 4$  SYM transforms covariantly under the dual superconformal group. Furthermore, it was shown in the same paper that the supercoefficients, which appear in the expansion of planar, one-loop amplitudes in a basis of box functions, transform covariantly under the symmetry, *i.e.* exactly in the same way as superamplitudes.

Dual conformal symmetry was first observed in the context of the duality between MHV scattering amplitudes and Wilson loops [3–5]. Strong indications of this duality were discovered in string theory in [3], where the calculation of scattering amplitudes at strong coupling was mapped to that of a Wilson loop with a particular polygonal contour which can be constructed by gluing together the null momenta of the scattered particles, following the order of the insertions of the string vertex operators on the worldsheet. Quite surprisingly, several calculations in perturbative  $\mathcal{N} = 4$  SYM, first at one [4, 5] and then at two loops [6–9], showed that the same duality holds also at weak coupling, with perfect agreement found between the perturbative Wilson loop and the MHV scattering amplitudes of the  $\mathcal{N} = 4$  theory computed in [10–13]. The perturbative Wilson loop/amplitude duality was recently studied in [14], where a numerical calculation of Wilson loops at two loops for an arbitrary number of particles was presented.

At strong coupling, the emergence of dual superconformal symmetry was understood in [15, 16] using a peculiar T-duality of the superstring theory on  $AdS_5 \times S^5$ , which combines bosonic [3] and fermionic T-duality transformations. The combined effect of these T-dualities maps the original string sigma model into a dual sigma model identical to the original one. More importantly, the T-duality also exchanges the original with the dual superconformal symmetries.

At weak coupling, dual conformal symmetry emerged as the ordinary conformal symmetry of the Wilson loop, which acts in the conventional way on 't Hooft's region momenta  $x_i$ . These are defined via the relations  $p_{i,\alpha\dot{\alpha}} = (x_i - x_{i+1})_{\alpha\dot{\alpha}}$ , where  $i = 1, \dots, n$ ,  $n$  is the number of scattered particles, and the identification  $x_{n+1} = x_1$  enforces momentum conservation. Dual conformal symmetry is broken by loop effects and the corresponding anomalous Ward identity for the Wilson loop was derived in [6, 7]. In particular, it was shown in [7] that the ABDK/BDS ansatz [13, 17] for the all-loop MHV amplitudes in  $\mathcal{N} = 4$  SYM is a solution to this anomalous Ward

identity.

The origin of the dual conformal anomaly at the quantum level can be traced to the presence of cusps in the polygonal contour of the Wilson loop. For a smooth contour dual conformal transformations would be an exact symmetry, however the cusps give rise to short-distance singularities which need to be regularised, and, hence, generate an anomaly in the dual conformal transformations at the loop level. These ultraviolet divergences are mapped to the conventional infrared divergences of the scattering amplitudes [18–25], thus suggesting an intimate link between infrared singularities and the dual conformal anomaly.

Inspired by the anomaly derived from the Wilson loop side and the duality with MHV amplitudes, dual conformal symmetry was extended in [1] to dual superconformal symmetry, acting on superamplitudes [26] defined in a dual on-shell superspace. Moreover, it was suggested that any superamplitude factorises naturally into the MHV superamplitude and a dual superconformal invariant factor  $\mathcal{R}$  as  $\mathcal{A} = \mathcal{A}_{\text{MHV}} \mathcal{R}$ . The MHV superamplitude factor completely encapsulates the anomaly, which is therefore a universal quantity. This remarkable conjecture was checked at one loop in [27] for the next-to-MHV (NMHV) superamplitudes up to nine particles. Very recently in [28, 29] dual conformal covariance was proved for one-loop NMHV superamplitudes with an arbitrary number of external particles.

The goal of this paper is to prove the dual conformal anomaly for generic (non-MHV) one-loop superamplitudes in the  $\mathcal{N} = 4$  theory. In order to do so, we will build on the results of [28], where the most generic expression for the dual conformal anomaly of all  $\mathcal{N} = 4$  superamplitudes was derived using only the result that the superamplitude can be expanded in terms of box functions [10]. The result of that calculation, reviewed in Section 2, was found to be the sum of two terms. The first one is precisely the one-loop anomaly conjectured in [1]. Therefore, the additional term must vanish if the conjecture of [1] is correct. This indeed happens for all MHV and NMHV superamplitudes, as was proved in [28] by using the explicit forms of these amplitudes derived in [10] and in [27]. Moreover, a new set of equations for the one-loop supercoefficients of a generic non-MHV amplitude were derived in [28] by assuming the vanishing of this additional term. In this paper we will prove that this term does indeed vanish for generic superamplitudes, thus providing a proof of the dual conformal anomaly conjectured in [1] at the one-loop level and, consequently, of the conformal equations presented in [28].

As will be explained in Section 2, this additional term is finite in four dimensions and can be written as a particular linear combination of two-mass triangle functions, which depend on multi-particle as well as two-particle invariants. On the other hand, the dual conformal anomaly of [1] diverges as  $1/\epsilon$  as  $\epsilon \rightarrow 0$ , and depends only on two-particle invariants through one-mass triangles. Note that we work here in di-

mensional regularisation with  $D = 4 - 2\epsilon$ . The different analytical structures of the two terms in the anomaly suggest that it is sufficient to study the discontinuities of the anomaly in all possible kinematic channels in order to prove that the additional term in the anomaly is in fact absent. Importantly, these discontinuities can be expressed in terms of appropriate phase space integrals. In Section 3 we will begin by calculating two-particle cuts of the one-loop anomaly for a generic superamplitude which are associated to discontinuities in multi-particle channels.<sup>1</sup> We will find that for any superamplitude these multi-particle discontinuities give rise to finite phase space integrals multiplied by  $\epsilon$ . Hence, all the multi-particle discontinuities of the dual conformal variation of a superamplitude vanish in four dimensions. With this result we can rule out any additional terms to the anomaly conjectured in [1]. We emphasise that our proof is general and applies to superamplitudes with arbitrary total helicity and an arbitrary number of external particles.

We have mentioned earlier a potential link between the dual conformal anomaly and infrared divergences. In Section 4 we expose this connection further by considering two-particle cuts of the anomaly in two-particle channels. Unlike the multi-particle discontinuities discussed above, the two-particle discontinuities of the anomaly are non-zero and finite in four dimensions. By uplifting the cut to a full loop diagram, akin to a procedure introduced in [31] for the calculation of splitting amplitudes, we calculate its leading infrared divergence, which in this case is of the order  $1/\epsilon$ . This turns out to reproduce precisely the anomaly of [1].

Our treatment of the two-particle channels exposes the leading  $1/\epsilon^2$  infrared singularity in this uplifted one-loop integral (which is further multiplied by one power of  $\epsilon$  from an anomalous Jacobian), and in principle could miss subleading  $1/\epsilon$  contributions to it; these, in turn, would lead to finite, unwanted contributions to the anomaly. However, we will argue that, thanks to the no-triangle and bubble property of one-loop amplitudes in  $\mathcal{N} = 4$  SYM [10], our approximation in fact captures all the infrared divergences of the above mentioned loop integral. This result, together with the absence of discontinuities of the anomaly in multi-particle channels, will provide us with a second (albeit intimately related) proof of the dual conformal anomaly conjectured in [1].

This second proof has the virtue of making more manifest the connection of the dual conformal anomaly of a generic scattering amplitude to its infrared divergences. We stress that a crucial ingredient of both proofs is the maximal supersymmetry of the theory. In the second proof, this enters directly through the no-triangle and bubble property of the  $\mathcal{N} = 4$  amplitudes [10]; in the first proof, it enters through the specific form of the most general anomaly, derived in [28].<sup>2</sup>

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<sup>1</sup>This has been done in the recent paper [30] for MHV amplitudes.

<sup>2</sup>This form of the anomaly can itself be thought of as a non-trivial consequence of the no triangle and bubble property.

## 2 Background

In this section we will first describe the structure of one-loop superamplitudes, and will then discuss the general form of the dual conformal anomaly.

### 2.1 One-loop superamplitudes

Scattering amplitudes in  $\mathcal{N} = 4$  SYM with a fixed number of particles and total helicity are naturally combined into superamplitudes [26]. These are defined in an on-shell superspace where to each particle  $i$  one associates the momentum  $p_i = \lambda_i \tilde{\lambda}_i$ , as well as a fermionic variables  $\eta_i^A$ , where  $A = 1, \dots, 4$  is an  $SU(4)$  index. The superamplitude can then be expanded in powers of the  $\eta_i^A$ 's, and each term of this expansion corresponds to a particular amplitude in  $\mathcal{N} = 4$  SYM with a fixed total helicity  $h_{\text{tot}} = \sum_{i=1}^n h_i$ . A term containing  $m_i$  powers of  $\eta_i$  corresponds to a scattering process where the  $i^{\text{th}}$  particle has helicity  $h_i = 1 - m_i/2$ . For instance, the MHV superamplitude is given by the following compact expression

$$\mathcal{A}_{\text{MHV}} = i (2\pi)^4 \frac{\delta^{(4)}(P) \delta^{(8)}(\Lambda)}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}, \quad (2.1)$$

where  $P := \sum_{i=1}^n \lambda_i \tilde{\lambda}_i$  and  $\Lambda := \sum_{i=1}^n \eta_i \lambda_i$  are the total momentum and supermomentum, respectively.

One-loop amplitudes in the maximally supersymmetric  $\mathcal{N} = 4$  theory can be expanded in a known basis of integrals which contains only box functions<sup>3</sup>  $F_i$ , and no triangle or bubble functions [10]. The functions  $F_i$  are related to the scalar box integrals  $I_i$  by a kinematic prefactor as follows. We call  $K_1, K_2, K_3$  and  $K_4$  the external momenta at the four corners of a given box function, which are expressed as sums of momenta  $p_i$  of external particles. The momenta  $K_{1\dots 4}$  can also be written in terms of the region momenta  $x_{1\dots 4}$ , e.g.  $K_1 = x_{12}$ , where  $x_{ij} := x_i - x_j$  (see Figure 1). Then, up to a numerical constant, the relation between the  $F$ 's and the  $I$ 's is

$$\begin{aligned} I_i &= -2 \frac{F_i}{\sqrt{R_i}}, \\ R_i &= (x_{13}^2 x_{24}^2)^2 - 2x_{13}^2 x_{24}^2 x_{12}^2 x_{34}^2 - 2x_{13}^2 x_{24}^2 x_{23}^2 x_{41}^2 + (x_{12}^2 x_{34}^2 - x_{23}^2 x_{41}^2)^2. \end{aligned} \quad (2.2)$$

Four-mass boxes are special from the point of view of the dual conformal symmetry, as they are infrared finite and invariant under the symmetry. We can then simplify the expression for  $\sqrt{R}$  to

$$\sqrt{R} \rightarrow x_{13}^2 x_{24}^2 - x_{23}^2 x_{41}^2, \quad (2.3)$$

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<sup>3</sup>We use a collective index  $i$  to denote the box function with external momenta  $K_{1\dots 4}$ .

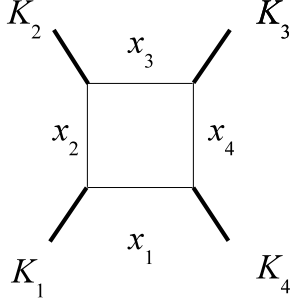


Figure 1: A generic box function.  $K_1, K_2, K_3$  and  $K_4$  denote the external momenta, and  $x_1, x_2, x_3$  and  $x_4$  the corresponding region momenta, with  $K_i = x_i - x_{i+1}$ ,  $i = 1, \dots, 4$ .

valid for all box functions except four-mass ones in the case where either  $x_{12}^2$  or  $x_{34}^2$  vanish. Notice that, under dual conformal inversions, one has

$$\sqrt{R_i} \rightarrow \frac{\sqrt{R_i}}{x_1^2 x_2^2 x_3^2 x_4^2}. \quad (2.4)$$

We expand a generic  $n$ -point one-loop superamplitude  $\mathcal{A}_n^{1\text{-loop}}$  in terms of box functions [10] as

$$\mathcal{A}_n^{1\text{-loop}} = \sum_{\{i,j,k,l\}} c(i,j,k,l) F(i,j,k,l), \quad (2.5)$$

where  $i, j, k, l$ , denote the four region momenta of the box function (as in Figure 1, with the labels 1, 2, 3, 4, replaced by  $i, j, k, l$ ).

In [2] it was shown that the supercoefficients  $c(i,j,k,l)$  transform covariantly under the symmetry. In order to deal with quantities which are invariant under dual conformal transformations rather than covariant, it is convenient to redefine the dual conformal generator  $K^\mu$  as [32]

$$K^\mu \rightarrow \hat{K}^\mu := K^\mu - 2 \sum_{i=1}^n x_i^\mu. \quad (2.6)$$

The covariance of the one-loop supercoefficients is then re-expressed as

$$\hat{K}^\mu c(i,j,k,l) = 0. \quad (2.7)$$

## 2.2 The structure of the anomaly

Dual conformal symmetry is violated at the quantum level by the presence of infrared divergences. In [28], using the expansion (2.5) of a generic superamplitude in a basis of boxes, together with the covariance of the one-loop supercoefficients (2.7), the dual conformal anomaly of an arbitrary superamplitude was written as

$$\hat{K}^\mu \mathcal{A}^{1\text{-loop}} = \sum_{\{i,j,k,l\}} c(i,j,k,l) K^\mu F(i,j,k,l) . \quad (2.8)$$

After calculating the various box anomalies  $K^\mu F(i,j,k,l)$ , (2.8) takes the form [28]

$$\begin{aligned} \hat{K}^\mu \mathcal{A}_n^{1\text{-loop}} &= 4\epsilon \mathcal{A}_n^{\text{tree}} \sum_{i=1}^n x_{i-1}^\mu x_{ii-2}^2 J(x_{ii-2}^2) \\ &\quad - 2\epsilon \sum_{i=1}^n \sum_{k=i+2}^{i+n-3} \mathcal{E}(i,k) \left[ x_{i-1}^\mu x_{ik}^2 - x_i^\mu x_{i-1k}^2 \right] J(x_{ik}^2, x_{i-1k}^2) , \end{aligned} \quad (2.9)$$

where

$$\mathcal{E}(i,k) := \sum_{j=k+1}^{i+n-2} c(i,k,j,i-1) - \sum_{j=i+1}^{k-1} c(i,j,k,i-1) , \quad (2.10)$$

is a particular combination of supercoefficients. Furthermore, (2.10) is valid for  $i < k$ ; if  $i > k$ , then the variable  $k$  appearing in the summation ranges of (2.10) has to be replaced by  $k+n$ . We have also introduced one-mass and two-mass triangle functions, defined as

$$J(a) := \frac{r_\Gamma}{\epsilon^2} (-a)^{-\epsilon-1} , \quad (2.11)$$

$$J(a,b) := \frac{r_\Gamma}{\epsilon^2} \frac{(-a)^{-\epsilon} - (-b)^{-\epsilon}}{(-a) - (-b)} , \quad (2.12)$$

respectively, where  $r_\Gamma := \Gamma(1+\epsilon)\Gamma^2(1-\epsilon)/\Gamma(1-2\epsilon)$ .

Equation (2.9) gives the most general expression for the anomaly of a one-loop superamplitude in  $\mathcal{N} = 4$  SYM with an arbitrary total helicity. In order to set the scene for our proof, let us now highlight the main characteristics of (2.9).

To begin with, the first term of (2.9) precisely matches the anomaly conjectured in [1]. Furthermore, it contains only one-mass triangles whose arguments are two-particle invariants. These triangles, multiplied by  $\epsilon$ , give rise to terms which diverge as  $1/\epsilon$  as  $\epsilon \rightarrow 0$ .

On the other hand, the second line of (2.9) contains two-mass triangles whose arguments can be two- or multi-particle invariants. In general, there is no two-mass triangle that has only two-particle invariants, except at five points, where amplitudes are only MHV or anti-MHV. This specific case has already been addressed explicitly in [1, 28] where it was shown that the anomaly of [1] is correctly reproduced.

The presence of multi-particle invariants in the second line of (2.9) is its key signature, and in the next section we will use the analyticity properties of this expression to prove that, in fact, this term identically vanishes. As a byproduct, this implies the conformal equations

$$\mathcal{E}(i, k) = 0, \quad i = 1, \dots, n, \quad k = i + 2, \dots, i + n - 3, \quad (2.13)$$

relating box coefficients, where  $\mathcal{E}(i, k)$  are given in (2.10). These relations were conjectured in [28] to hold for any superamplitude, and checked explicitly for the infinite sequences of MHV and NMHV superamplitudes. They can be solved to give expressions for all one-mass, two-mass easy and half of the two-mass hard box coefficients in terms of the remaining box coefficients.

### 3 The first proof

We perform the proof of the one loop dual conformal anomaly in two steps:

**1.** We will calculate the discontinuities of the anomaly using conventional unitarity [33], and prove that for multi-particle channels, the result for the discontinuity is given by  $\epsilon$  times an integral which is finite in four dimensions. The result for such a discontinuity therefore vanishes in four dimensions.

**2.** We will calculate the discontinuity of the anomaly in a multi-particle channel directly from (2.9), and impose that this vanishes. This precisely implies the dual conformal equations (2.13) and therefore proves the form of the anomaly conjectured in [1] for all one-loop superamplitudes in the  $\mathcal{N}=4$  theory,

$$\hat{K}^\mu \mathcal{A}_n^{1\text{-loop}} = 4\epsilon \mathcal{A}_n^{\text{tree}} \sum_{i=1}^n x_{i-1}^\mu x_{ii-2}^2 J(x_{ii-2}^2). \quad (3.1)$$

We now proceed directly to the proof.

**1.** Consider the discontinuity of the superamplitude in a certain multi-particle channel  $P_L^2$ . We wish to show that this is conformally invariant (this is not true



for the two-particle channel – such cuts will be considered in the following section). The corresponding cut diagram is represented in Figure 2, and is expressed by the following phase space integral:

$$\int d\mu_{i,\dots,j} \mathcal{A}_L(l_2, l_1, i, \dots, j) \mathcal{A}_R(-l_1, -l_2, j+1, \dots, i-1) . \quad (3.2)$$

The integration measure is defined as

$$d\mu_{i,\dots,j} := d\text{LIPS}(l_2, l_1; P_L) d^4\eta_{l_1} d^4\eta_{l_2} \delta^{(8)}(\eta_{l_1} \lambda_{l_1} + \eta_{l_2} \lambda_{l_2} + \Lambda_L) , \quad (3.3)$$

where

$$d\text{LIPS}(l_2, l_1; P_L) = d^D l_1 d^D l_2 \delta^{(+)}(l_1^2) \delta^{(+)}(l_2^2) \delta^{(D)}(l_1 + l_2 + P_L) , \quad (3.4)$$

is the phase space measure, and  $d^4\eta_{l_1} d^4\eta_{l_2} \delta^{(8)}(\eta_{l_1} \lambda_{l_1} + \eta_{l_2} \lambda_{l_2} + \Lambda_L)$  is the fermionic integration measure. We have defined

$$P_L := \sum_{k=i}^j \lambda_k \tilde{\lambda}_k , \quad \Lambda_L := \sum_{k=i}^j \eta_k \lambda_k , \quad (3.5)$$

to be the total momenta and supermomenta flowing out of the left hand side of the cut diagram. In (3.2) we are omitting overall delta functions imposing momentum and supermomentum conservation.

In the appendix we compute the dual conformal transformation of the discontinuity in the  $x_{ij+1}^2$  channel, with the result

$$\text{disc}_{x_{ij+1}^2} [\hat{K}^\mu \mathcal{A}_n^{1-\text{loop}}] = 2(4-D) \int d^D y \delta^{(+)}((y-x_i)^2) \delta^{(+)}((x_{j+1}-y)^2) \left[ y^\mu \langle l_1 l_2 \rangle^4 \mathcal{A}_L \mathcal{A}_R \right] . \quad (3.6)$$

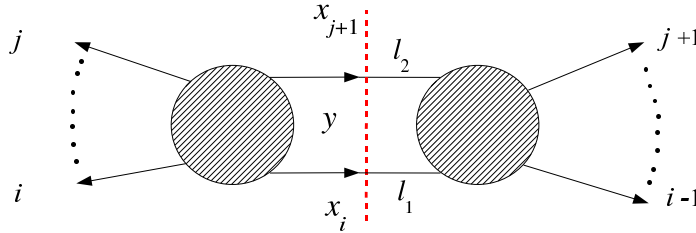


Figure 2: A cut diagram reproducing the discontinuity of the anomaly for a generic superamplitude in a kinematic channel  $x_{ij+1}^2$ . When  $x_{ij+1}^2$  is a multi-particle invariant, the phase space integral corresponding to this cut diagram is finite, and vanishes in four dimensions due to the factor of  $D-4$  on the right hand side of (3.6).

In order to understand whether (3.6) leads to a contribution to the anomaly, we analyse the singularities of the integral in that equation. To this end, we first consider

the phase space integral giving the discontinuity of the superamplitude in the same channel  $x_{ij+1}^2$ . This is given by

$$\text{disc}_{x_{ij+1}^2} \mathcal{A}_n^{1\text{-loop}} = \int d^D y \delta^{(+)}((y - x_i)^2) \delta^{(+)}((x_{j+1} - y)^2) \left[ \langle l_1 l_2 \rangle^4 \mathcal{A}_L \mathcal{A}_R \right]. \quad (3.7)$$

As is well known, there is a crucial distinction in the infrared properties of (3.7) between the cases when the channel is a multi-particle or a two-particle one. When  $x_{ij+1}^2$  is a multi-particle channel, the integral appearing on the right hand side of (3.7) is free of infrared divergences and hence can be calculated in four dimensions, see [34] and [31] for a discussion of this point.<sup>4</sup> This is of course in agreement with the general expression of the infrared divergences of one-loop amplitudes in  $\mathcal{N} = 4$  SYM, given by [35]

$$\mathcal{A}_n^{1\text{-loop}}|_{\text{IR}} = -r_\Gamma \mathcal{A}_n^{\text{tree}} \sum_{i=1}^n \frac{(-x_{ii+2}^2)^{-\epsilon}}{\epsilon^2}, \quad (3.8)$$

which only contain two-particle invariants formed with adjacent momenta (but no multi-particle invariant).

We now make the observation that the integral we are really interested in, namely that on the right hand side of (3.6) is very similar to the integral appearing in (3.7). More precisely, (3.6) contains an extra power of  $2(D - 4)$  and a  $y^\mu$  in the integrand compared to (3.7). The presence in (3.6) of  $y = x_{j+1} - l_1$  by itself cannot lead to any infrared singularity (the term containing  $l_1$  will only give rise to terms which are better behaved in the infrared). Since (3.7) is infrared finite, we conclude that the presence of a factor of  $D - 4$  multiplying the discontinuity of the anomaly (3.6) will make the result vanish. Hence, for a generic multi-particle channel  $x_{ij+1}^2$ ,

$$\text{disc}_{x_{ij+1}^2} [\hat{K}^\mu \mathcal{A}_n^{1\text{-loop}}] = 0, \quad j \neq i + 1. \quad (3.9)$$

This concludes the first part of the proof. Notice that (3.9) is in agreement with [30], where it was observed that the discontinuities of the one-loop MHV amplitude in multi-particle channels, calculated in [34], are dual conformal invariant. Our result (3.9) is however completely general, in that it applies to all one-loop amplitudes, including non-MHV.

**2.** We now wish to use the absence of conformal anomalies in the multi-particle cut (3.9) to constrain the expression (2.9). More precisely, we will use the fact that in each of the  $n(n - 5)/2$  multi-particle channels the discontinuity of the anomaly

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<sup>4</sup>For the sake of this first proof, we are only interested in multi-particle channels, as discussed above. We will later on discuss the two-particle channel discontinuities, to show how the anomaly arises precisely from such singular channels.

vanishes in order to prove the conformal equations (2.13), and hence the form of the conformal anomaly for generic amplitudes.

To this end, we focus on the terms on the right hand side of (2.9) which have a discontinuity in a certain multi-particle channel  $x_{ik}^2$ . There are four such terms:

$$\begin{aligned}
\hat{K}^\mu \mathcal{A}_n^{1\text{-loop}} &\ni -2\epsilon \mathcal{E}(i, k) \left[ x_{i-1}^\mu x_{ik}^2 - x_i^\mu x_{i-1k}^2 \right] J(x_{ik}^2, x_{i-1k}^2) \\
&- 2\epsilon \mathcal{E}(i+1, k) \left[ x_i^\mu x_{i+1k}^2 - x_{i+1}^\mu x_{ik}^2 \right] J(x_{i+1k}^2, x_{ik}^2) \\
&- 2\epsilon \mathcal{E}(k, i) \left[ x_{k-1}^\mu x_{ik}^2 - x_k^\mu x_{k-1i}^2 \right] J(x_{ik}^2, x_{k-1i}^2) \\
&- 2\epsilon \mathcal{E}(k+1, i) \left[ x_k^\mu x_{k+1i}^2 - x_{k+1}^\mu x_{ik}^2 \right] J(x_{k+1i}^2, x_{ik}^2) .
\end{aligned} \tag{3.10}$$

The last two lines are obtained from the first two by simply exchanging  $i$  with  $k$ . The discontinuity of a triangle function is given by

$$\text{disc}_b [\epsilon J(a, b)] = \frac{2\pi i}{b-a} + \mathcal{O}(\epsilon) , \tag{3.11}$$

therefore

$$\begin{aligned}
&\text{disc}_{x_{ik}^2} [K^\mu \mathcal{A}_n^{1\text{-loop}}] \\
&= 2\pi i \left[ \mathcal{E}(i, k) \frac{x_{i-1}^\mu x_{ik}^2 - x_i^\mu x_{i-1k}^2}{x_{ik}^2 - x_{i-1k}^2} + \mathcal{E}(i+1, k) \frac{x_i^\mu x_{i+1k}^2 - x_{i+1}^\mu x_{ik}^2}{x_{ik}^2 - x_{i+1k}^2} \right. \\
&\quad \left. \mathcal{E}(k, i) \frac{x_{k-1}^\mu x_{ik}^2 - x_k^\mu x_{k-1i}^2}{x_{ik}^2 - x_{k-1i}^2} + \mathcal{E}(k+1, i) \frac{x_k^\mu x_{k+1i}^2 - x_{k+1}^\mu x_{ik}^2}{x_{ik}^2 - x_{k+1i}^2} \right] \\
&= 0 ,
\end{aligned} \tag{3.12}$$

where in the last step we have used (3.9).

Equation (3.12) is a vector equation, which we can rewrite as

$$\mathcal{E}(i, k) v_{ik}^\mu + \mathcal{E}(i+1, k) v_{i+1k}^\mu + \mathcal{E}(k, i) v_{ki}^\mu + \mathcal{E}(k+1, i) v_{k+1i}^\mu = 0 , \tag{3.13}$$

where

$$v_{ik}^\mu := \frac{x_{i-1}^\mu x_{ik}^2 - x_i^\mu x_{i-1k}^2}{x_{ik}^2 - x_{i-1k}^2} , \tag{3.14}$$

and we remark that the four vectors  $v_{ik}^\mu$ ,  $v_{i-1k}^\mu$ ,  $v_{ki}^\mu$  and  $v_{k-1i}^\mu$  are in general linearly independent in four dimensions. Hence we conclude that the coefficients  $\mathcal{E}(i, k)$ ,  $\mathcal{E}(k, i)$ ,  $\mathcal{E}(i+1, k)$ ,  $\mathcal{E}(k+1, i)$  must vanish independently.

This proves the conformal equations (2.13), and therefore we conclude that the dual conformal anomaly for an arbitrary (non-MHV) amplitude is given by (3.1).

This completes our proof. We conclude this section with a couple of additional comments.

First, we have managed to prove  $n(n-4)$  conformal equations  $\mathcal{E}(i, k)=0$  in (2.13) from considering just  $n(n-5)/2$  multi-particle cuts of the amplitude. We can do this since each multiparticle cut anomaly is a vectorial equation and hence gives four independent conditions, thus we really obtain  $2n(n-5)$  (dependent) conditions. These are precisely the conditions (3.13). Each multiparticle cut  $x_{ik}^2$  leads to the condition  $\mathcal{E}(i, k) = 0$  and  $\mathcal{E}(k, i) = 0$ . This leaves only the ‘boundary case’  $\mathcal{E}(i, i+2) = 0$  potentially unaccounted for. Fortunately (3.13) also gives  $\mathcal{E}(i+1, k) = 0$  which for  $k = i+3$  gives us precisely this boundary case with the (arbitrary) label  $i$  shifted to  $i+1$ .

Second, we mention an important point which could have affected our proof. It has been recently pointed out in [30, 36] that some of the dual superconformal generators do not precisely annihilate the superamplitude but leave behind a delta-function supported contribution of the type of an holomorphic anomaly [37]. It is important for our proof that there is no holomorphic anomaly for the special conformal generator  $K^\mu$  we are interested in. Indeed, had  $K^\mu$  acting on the tree-level amplitudes  $\mathcal{A}_L$  or  $\mathcal{A}_R$  in (3.7) produced delta-function contributions, the phase space integration in (3.7) would be localised, and new, unwanted contributions to the dual conformal anomaly would be generated.

Fortunately, the absence of holomorphic anomaly contributions to the dual special conformal generator  $K^\mu$  has been shown for all tree-level amplitudes in [36], and specifically for MHV superamplitudes and six-point NMHV tree-level superamplitudes in [30]. We recall that the holomorphic anomaly arises from

$$\frac{\partial}{\partial \tilde{\lambda}^{\dot{\alpha}}} \frac{1}{\langle \lambda \mu \rangle} = 2\pi \tilde{\mu}_{\dot{\alpha}} \delta(\langle \lambda \mu \rangle) \delta([\tilde{\lambda} \tilde{\mu}]) . \quad (3.15)$$

In order to get a contribution from holomorphic anomalies which would lead to a localisation of the phase space integral in (3.7), one should then identify all possible physical singularities in the scattering amplitude of the type  $1/\langle lk \rangle$ , where  $l$  is any one of the cut loop momenta, and  $k$  one of the legs which are adjacent to it.<sup>5</sup> At tree level, such singularities arise only from collinear kinematics, and in [30] it has been shown that in the action of  $K_{\alpha\dot{\alpha}}$  on  $1/\langle ii+1 \rangle$ , the holomorphic anomaly contributions cancel. Therefore, the special conformal generator is not affected by holomorphic anomalies.<sup>6</sup>

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<sup>5</sup>We consider colour-ordered amplitudes, hence such singularities can only involve particles that are adjacent in colour space.

<sup>6</sup>Unlike the dual special conformal generator, the dual supersymmetry generator  $\bar{Q}$  does suffer from a holomorphic anomaly [30].

## 4 Unearthing the anomaly with two-particle cuts

To complete our discussion, we now address the two-particle channel cuts of the conformal variation of the amplitudes in more detail. This will reveal the close relation between infrared divergences of the amplitude and the dual conformal anomaly. For

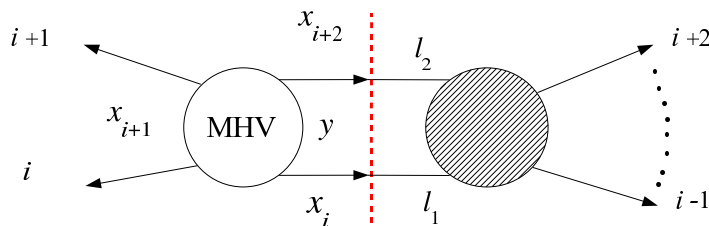


Figure 3: *The cut diagram reproducing the discontinuity of the anomaly for a generic superamplitude in the two-particle channel  $s_{ii+1} = x_{ii+2}^2$ . In this case the superamplitude on the left hand side has four particles and, hence, must be a MHV superamplitude. This diagram has an infrared divergence arising from the region of integration where  $l_1 \sim -p_i$  and  $l_2 \sim -p_{i+1}$ . In this region we also have that  $y \sim x_{i+1}$  where  $y$  is the region momentum between the two cut legs.*

concreteness let us consider the two-particle channel cut of a generic one-loop amplitude represented in Figure 3. This cut gives rise to the following phase space integral,

$$\text{disc}_{x_{ii+2}^2} \mathcal{A}_n^{1\text{-loop}} = \int d\mu_{ii+1} \mathcal{A}_{\text{MHV}}(i, i+1, l_2, l_1) \mathcal{A}_R(-l_1, -l_2, i+2, \dots, i-1), \quad (4.1)$$

while the corresponding cut diagram of the dual conformal anomaly is given by

$$\text{disc}_{x_{ii+2}^2} [K^\mu \mathcal{A}_n^{1\text{-loop}}] = 4\epsilon \int d\mu_{ii+1} \left[ y^\mu \mathcal{A}_{\text{MHV}}(i, i+1, l_2, l_1) \mathcal{A}_R(-l_1, -l_2, i+2, \dots, i-1) \right], \quad (4.2)$$

where the integration measure appearing in both expressions is defined in (3.3).

The phase space integrals appearing in (4.1) and (4.2) are both infrared divergent. The divergence arises from a region where  $l_1$  becomes collinear to  $p_i$  and  $l_2$  becomes collinear to  $p_{i+1}$  at the same time [31]. This occurrence of simultaneous collinear singularities is special to two-particle cuts and is necessary to produce the expected infrared divergences. For generic kinematics, *i.e.* if  $p_i$  and  $p_{i+1}$  are not collinear, momentum conservation of the four-point MHV amplitude implies that the singular region of the momentum integral is confined to

$$l_1 \rightarrow -p_i, \quad l_2 \rightarrow -p_{i+1}. \quad (4.3)$$

In this singular region of loop momentum space, the region momentum  $y$  localises on the region momentum  $x_{i+1}$ , which will be of importance in the following.

In the following we will show that the exact one-loop anomaly and the infrared divergent part of the amplitude can be extracted by using an approximation which focuses exactly on this peculiar loop momentum region. A couple of explanations are in order here:

- We observe that the leading infrared singularity of both integrals is correctly captured by replacing  $-l_1$  and  $-l_2$  in the tree-level superamplitude  $\mathcal{A}_R$  by  $p_i$  and  $p_{i+1}$ , respectively. Importantly, this allows us to pull  $\mathcal{A}_R$  out in front of the integrals (4.1) and (4.2). Note that this tree-level superamplitude has the same total helicity as the one-loop amplitude under consideration. Furthermore, in the integrand of (4.2) we can replace  $y$  with  $x_{i+1}$  and factor it out of the integral as well. This factorisation property of the leading infrared singularity is illustrated in Figure 4.
- We know from (3.9) in the previous section that the anomaly does not have discontinuities in multi-particle channels and, hence, it can only depend on two-particle invariants.
- Our approximation of the cut integrals (4.1) and (4.2) removes all dependence on momentum invariants other than the two-particle invariant  $x_{ii+2}^2$ . This allows us to directly uplift the cut integrals to full loop integrals, *i.e.* replace the two on-shell  $\delta$ -functions in the cut by propagators, very much as was done in [31] for splitting amplitudes.
- The remaining issue is to rule out any subleading infrared terms that our approximation might miss. For the amplitudes, this is easily settled by recalling that in  $\mathcal{N} = 4$  SYM all one-loop amplitudes are linear combinations of box functions and that all potentially infrared divergent terms are expressed in terms of one-mass triangle functions with a two-particle invariant argument, which behave as  $(-x_{ii+2}^2)^{-\epsilon}/\epsilon^2$ . Indeed, after uplifting the cut integral, one obtains one-mass triangle integrals with the expected coefficient. If we now apply dual special conformal transformation on our integrand, one power of the loop momentum appears in the numerator. Uplifting then gives rise to linear box functions, which can be reduced to scalar boxes and triangles. The result consists of finite terms and terms that are of the form  $(-s)^{-\epsilon}/\epsilon^2$  or  $(-t)^{-\epsilon}/\epsilon^2$  for two- (multi-)particle invariants  $s$  ( $t$ ). Crucially, there are no subleading  $1/\epsilon$  terms, *i.e.* bubble functions. Since the dual conformal transformation comes with a factor of  $\epsilon$ , only one-mass triangles with two-particle invariants survive. So in essence the no-bubble and no-triangle property of one-loop amplitudes in

$\mathcal{N} = 4$  SYM ensures that our procedure is valid.<sup>7</sup>

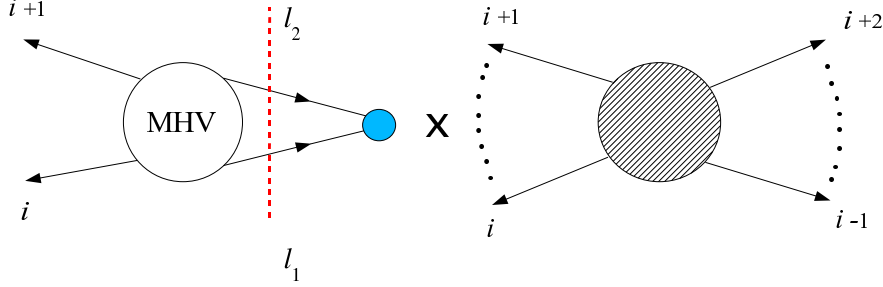


Figure 4: *Graphic representation of the factorised structure of the two-particle channel discontinuities of the amplitude and of the anomaly in (4.4) and (4.5).*

After these comments we proceed now to the explicit evaluation of the two-particle cut integrals (4.1) and (4.2). For the infrared divergent part of the amplitude we find

$$\begin{aligned} \left[ \mathcal{A}_n^{1\text{-loop}} \right]_{x_{ii+2}^2\text{-cut}} \Big|_{\text{IR}} &= \mathcal{A}_n^{\text{tree}} \int \frac{d^D y}{(2\pi)^D} \left[ \frac{\langle l_1 l_2 \rangle^4 \mathcal{A}_{\text{MHV}}(i, i+1, l_2, l_1)}{(x_i - y)^2 (x_{i+2} - y)^2} \right]_{x_{ii+2}^2\text{-cut}} , \\ &= \mathcal{A}_n^{\text{tree}} \int \frac{d^D y}{(2\pi)^D} \left[ \frac{x_{ii+2}^2}{(x_i - y)^2 (x_{i+1} - y)^2 (x_{i+2} - y)^2} \right]_{x_{ii+2}^2\text{-cut}} , \end{aligned} \quad (4.4)$$

where we have performed some spinor algebra and used (4.3) to simplify the numerator of the integrand to obtain the second line. We arrive at an expression that is proportional to a scalar one-mass triangle. Similarly, for the dual conformal anomaly of the amplitude we obtain

$$\begin{aligned} [K^\mu \mathcal{A}_n^{1\text{-loop}}]_{x_{ii+2}^2\text{-cut}} &= 4\epsilon x_{i+1}^\mu \mathcal{A}_n^{\text{tree}} \int \frac{d^D y}{(2\pi)^D} \left[ \frac{\langle l_1 l_2 \rangle^4 \mathcal{A}_{\text{MHV}}(i, i+1, l_2, l_1)}{(x_{i+2} - y)^2 (x_i - y)^2} \right]_{x_{ii+2}^2\text{-cut}} \\ &= 4\epsilon x_{i+1}^\mu \mathcal{A}_n^{\text{tree}} \int \frac{d^D y}{(2\pi)^D} \left[ \frac{x_{ii+2}^2}{(x_i - y)^2 (x_{i+1} - y)^2 (x_{i+2} - y)^2} \right]_{x_{ii+2}^2\text{-cut}} . \end{aligned} \quad (4.5)$$

We notice that these two relations establish a link between the universal infrared behaviour of the amplitudes and the dual conformal anomaly, since these imply that

$$[K^\mu \mathcal{A}_n^{1\text{-loop}}]_{x_{ii+2}^2\text{-cut}} = 4\epsilon x_{i+1}^\mu \left[ \mathcal{A}_n^{1\text{-loop}} \right]_{x_{ii+2}^2\text{-cut}} \Big|_{\text{IR}} .$$

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<sup>7</sup>For the precise form of the reduced linear boxes see (2.9). However in this section we wish to give general arguments without referring to this formula for the generic anomaly.

The expression (4.4) can be freely uplifted and summed over all two-particle channels (recalling that all multi-particle cuts vanish) to give us directly the well known expression for the infrared divergent part of the amplitude (3.8). Similarly, (4.5) gives directly

$$K^\mu \mathcal{A}_n^{1\text{-loop}} = 4\epsilon \mathcal{A}_n^{\text{tree}} \sum_{i=1}^n x_{i+1}^\mu x_{ii+2}^2 J(x_{ii+2}^2) . \quad (4.6)$$

The right hand side of (4.6) is nothing but the dual conformal anomaly. We have thus managed to link this anomaly to infrared-divergent two-particle channel cut diagrams. This provides a derivation of the form of the anomaly which exposes in a very direct manner the exact coefficient of the anomaly.

## Acknowledgements

It is a pleasure to thank Bill Spence for stimulating discussions. This work was supported by the STFC under a Rolling Grant ST/G000565/1. The work of PH is supported by an EPSRC Standard Research Grant EP/C544250/1. GT is supported by an EPSRC Advanced Research Fellowship EP/C544242/1 and by an EPSRC Standard Research Grant EP/C544250/1.

## A Conformal transformation of a cut diagram

We now consider the conformal variation of the discontinuity integral. This is given by the following expression,

$$\int d\mu'_{i,\dots,j} \mathcal{A}_L(l'_2, l'_1, i', \dots, j') \mathcal{A}_R(-l'_1, -l'_2, (j+1)', \dots, (i-1)') , \quad (\text{A.1})$$

where the prime means that the momenta have been replaced by their conformally varied expressions (we also freely replace the loop momenta and the measure by conformally varied expressions). We recall that momenta are written as differences of region momenta as

$$p_i := x_i - x_{i+1} , \quad (\text{A.2})$$

and that under an infinitesimal special conformal transformation one has

$$K^\nu x^\mu := \eta^{\mu\nu} x^2 - 2 x^\mu x^\nu . \quad (\text{A.3})$$



In spinor notation, (A.3) becomes

$$K^{\beta\dot{\beta}}x_i^{\alpha\dot{\alpha}} = -2x_i^{\alpha\dot{\beta}}x_i^{\beta\dot{\alpha}}, \quad (\text{A.4})$$

and one can easily check that the transformations

$$K^{\beta\dot{\beta}}\lambda_i^\alpha = -2(x_i^{\alpha\dot{\beta}}\lambda_i^\beta - \mu_i\lambda_i^\alpha\lambda_i^\beta\tilde{\lambda}_i^{\dot{\beta}}), \quad (\text{A.5})$$

$$K^{\beta\dot{\beta}}\tilde{\lambda}_i^{\dot{\alpha}} = -2(x_i^{\dot{\alpha}\beta}\tilde{\lambda}_i^{\dot{\beta}} - (1-\mu_i)\tilde{\lambda}_i^{\dot{\alpha}}\tilde{\lambda}_i^{\dot{\beta}}\lambda_i^\beta), \quad (\text{A.6})$$

are consistent with this (consider  $K^{\beta\dot{\beta}}(\lambda_i^\alpha\tilde{\lambda}_i^{\dot{\alpha}}) = K^{\beta\dot{\beta}}x_{i+1}^{\alpha\dot{\alpha}}$ ). The standard choice of spinor transformation [1] is obtained by simply setting  $\mu_i = 0$ . We will have to consider these general transformations involving the free parameter  $\mu_i$  in order to correctly cope with the transformation of the spinors associated with the loop momenta.

We have

$$K^\mu x_{ii+2}^2 = -2(x_i^\mu + x_{i+2}^\mu)x_{ii+2}^2, \quad (\text{A.7})$$

and

$$K^\mu \langle ii+1 \rangle = -2 \left[ x_i^\mu (1-\mu_i) + x_{i+1}^\mu (\mu_i - \mu_{i+1}) + x_{i+2}^\mu \mu_{i+1} \right] \langle ii+1 \rangle, \quad (\text{A.8})$$

$$K^\mu [ii+1] = -2 \left[ x_i^\mu \mu_i + x_{i+1}^\mu (-\mu_i + \mu_{i+1}) + x_{i+2}^\mu (1-\mu_{i+1}) \right] [ii+1]. \quad (\text{A.9})$$

After performing the fermionic integrations and one of the two momentum integrations, as well as a shift in the integration variable to  $y = l_1 + x_i$ , the cut superamplitude becomes

$$\int d^D y \delta^{(+)}(l_1^2) \delta^{(+)}(l_2^2) \langle l_1 l_2 \rangle^4 \mathcal{A}_L \mathcal{A}_R, \quad (\text{A.10})$$

where  $l_1 = \lambda_{l_1} \tilde{\lambda}_{l_1} = y - x_i$  and  $l_2 = \lambda_{l_2} \tilde{\lambda}_{l_2} = x_{j+1} - y$ .

The transformation of the various components of the cut diagram read as follows,

$$\begin{aligned} & K \mathcal{A}_L(l_2, l_1, i, \dots, j) \\ &= 2 \left[ \sum_{a=i+1}^j x_a (1 - 2\mu_a + 2\mu_{a-1}) + x_{j+1} (1 - 2\mu_{l_2} + 2\mu_j) \right. \\ & \quad \left. + y (1 - 2\mu_{l_1} + 2\mu_{l_2}) + x_i (1 - 2\mu_i + 2\mu_{l_1}) \right] \mathcal{A}_L(l_2, l_1, i, \dots, j), \end{aligned} \quad (\text{A.11})$$

and

$$\begin{aligned} & K \mathcal{A}_R(-l_1, -l_2, j+1, \dots, i-1) \\ &= 2 \left[ \sum_{b=j+2}^{i-1} x_b (1 - 2\mu_b + 2\mu_{b-1}) + x_i (1 - 2\mu'_{l_1} + 2\mu_{i-1}) \right. \\ & \quad \left. + y (1 - 2\mu'_{l_2} + 2\mu'_{l_1}) + x_{j+1} (1 - 2\mu_{j+1} + 2\mu'_{l_2}) \right] \mathcal{A}_R(-l_1, -l_2, j+1, \dots, i-1). \end{aligned} \quad (\text{A.12})$$

Furthermore, one has

$$K \delta^{(+)}(l_1^2) = 2(x_i + y) \delta^{(+)}(l_1^2) , \quad (\text{A.13})$$

$$K \delta^{(+)}(l_2^2) = 2(x_{j+1} + y) \delta^{(+)}(l_2^2) , \quad (\text{A.14})$$

$$K d^D y = -2D y d^D y . \quad (\text{A.15})$$

Here the  $\mu'$  parameters are defined as  $\mu'_{l_1} := 1 - \mu_{l_1}$  and  $\mu'_{l_2} := 1 - \mu_{l_2}$ , accounting for the fact that, in  $\mathcal{A}_R$ ,  $l_1$  and  $l_2$  are in the reverse cyclic ordering to  $\mathcal{A}_L$ . The covariance of all tree-level amplitudes under conformal transformations was proved in [2] and indeed covariance under the more general transformations defined here was also proven there.

Putting all this together we get that the cut amplitude transforms with weight

$$2 \sum_{a=1}^n x_a^\mu (1 - 2\mu_a + 2\mu_{a-1}) + 2(4 - D)y^\mu . \quad (\text{A.16})$$

The first term is the expected covariant term, whereas the second is the contribution to the one-loop dual conformal anomaly quoted in (3.6).

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